Compositions of Boolean functions with monotone functions

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For a class $\mathcal{C}$ of Boolean functions, we say that a Boolean function f is a *$\mathcal{C}$-subfunction* of a Boolean function g, denoted $f \preceq_{\mathcal{C}} g$, if $f = g(h_1, \ldots, h_n)$, where all the inner functions h_i are members of $\mathcal{C}$. Functions f and g are *$\mathcal{C}$-equivalent,* denoted $f \equiv_{\mathcal{C}} g$ if they are $\mathcal{C}$-subfunctions of each other. The $\mathcal{C}$-subfunction relation $\preceq_{\mathcal{C}}$ is a preorder on the set Ω of all Boolean functions if and only if $\mathcal{C}$ is a clone. If $\mathcal{C}$ is a clone, then $\equiv_{\mathcal{C}}$ is an equivalence relation. It is natural to ask whether there is an infinite descending chain of $\mathcal{C}$-subfunctions and what is the size of the largest antichain of $\mathcal{C}$-incomparable functions.

In this presentation, we focus on the monotone clones of the Post lattice: the clone M of all monotone functions and the clones M_0, M_1, and M_c of 0-preserving, 1-preserving, and constant-preserving monotone functions, respectively.

For $\mathcal{C} = M, M_0, M_1, M_c$, we present a characterization of the $\mathcal{C}$-equivalence classes, based on a simple measure of alternation of the true and false points in the hypergraph representation of Boolean functions. We also show that the $\mathcal{C}$-subfunction relation is well-founded, i.e., there is no infinite descending chain of $\mathcal{C}$-subfunctions and no infinite antichain of $\mathcal{C}$-incomparable functions.